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Geometric second order field equations for general tensor gauge fields

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Abstract: Higher spin tensor gauge fields have natural gauge-invariant field equations written in terms of generalised curvatures, but these are typically of higher than second order in derivatives. We construct geometric second order field equations and actions for general higher spin boson fields, and first order ones for fermions, which are non-local but which become local on gauge-fixing, or on introducing auxiliary fields. This generalises the results of Francia and Sagnotti to all representations of the Lorentz group.

Keywords: Field Theories in Higher Dimensions, Gauge Symmetry, Differential and Algebraic Geometry.

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Contents

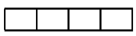
1. Introduction

Fields in higher spin representations of the Lorentz group arise in a variety of contexts. In perturbative string theory they are present in the spectrum of massive modes, and limits in which an infinite number of these become massless are of considerable interest, as they could lead to symmetric phases of string theory with an infinite-dimensional unbroken symmetry group. Interacting theories with infinite numbers of massless higher spin fields with anti-de Sitter vacua have been constructed in [1, 2], but so far these have only been constructed in anti-de Sitter spaces of dimensions $D \leq 5$, and in certain generalised spacetimes. These are associated with higher spin algebras, and higher spin superalgebras have recently been constructed in dimensions $D \leq 7$ [3]. The free covariant field theories for higher spin gauge fields have been discussed in [4]–[7]. Tensor gauge fields in unusual representations of the Lorentz group are also an inevitable consequence of dualising certain conventional gauge theories. This was analysed in [8], where the dual forms of linearised gravity were found in arbitrary dimensions and duality was discussed for general tensor fields.

Recently, there has been considerable progress in the understanding of free higher spin massless gauge fields. In [9, 10], covariant local free field equations and gauge-invariant actions were given for gauge fields in D dimensions transforming in any representation of $GL(D, \mathbb{R})$. The field equations were given in terms of field strengths so that they were

JHEP05(2003)019

manifestly invariant under higher spin gauge transformations. However, these field equations in general involved more than two derivatives of the gauge field. (The field equations were second order only for those gauge fields in representations corresponding to Young tableaux with no more than two columns.) In [11], second order field equations were considered for gauge fields in completely symmetric tensor representations. On fixing some of the symmetries, these reduced to the local second order field equations of Fronsdal [13]. However, the covariant form of the field equations of [11] were written in terms of invariant field strengths and were non-local, involving inverse D'Alembertian operators. These non-localities can be removed, however, by gauge-fixing, or by introducing auxiliary compensator fields. This gauge-fixing has been described in detail in [12] for totally symmetric tensors where it was found that they indeed describe the same number of on-shell degrees of freedom as those derived using the Fronsdal formalism. The purpose of this note is to write down geometric second order field equations for all representations by combining the two approaches. The idea is simple; consider one of the geometric higher derivative field equations in D dimensions of [9, 10], which is of order $2n$ in derivatives. The free field equation is the vanishing of a higher spin generalisation of the Einstein tensor, $E = 0$, where E involves $2n$ partial derivatives of the gauge field A. In a physical gauge (i.e. a transverse traceless gauge) this reduces to $\Box^n A = 0$ where $\Box := \partial_\mu \partial^\mu$. Then a suitable second order field equation generalising that of [11] is $\frac{1}{\Box^{n-1}} E = 0$, and in physical gauge this reduces to $\Box A = 0$, as it should. This means that the apparent non-locality of the equation can be eliminated by a suitable gauge choice, as in [11], and it can also be eliminated by introducing auxiliary fields [11]. An analogous procedure, again based on [11], is presented for obtaining first order field equations for spinor-valued fermionic fields in any representation of the Lorentz group.

Consider for example a fourth rank totally symmetric tensor gauge field $A_{\mu\nu\rho\sigma} = A_{(\mu\nu\rho\sigma)}$ represented by the Young tableau  which has the gauge transformation

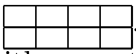
$$\delta A_{\mu\nu\rho\sigma} = 4\, \partial_{(\mu} \Lambda_{\nu\rho\sigma)} \tag{1.1}$$

with totally symmetric tensor gauge parameter $\Lambda_{\nu\rho\sigma}$. The gauge-invariant field strength is

$$F_{\mu_1\mu_2\nu_1\nu_2\rho_1\rho_2\sigma_1\sigma_2} = \partial_{\mu_1} \partial_{\nu_1} \partial_{\rho_1} \partial_{\sigma_1} A_{\mu_2\nu_2\rho_2\sigma_2} - (\mu_1 \leftrightarrow \mu_2) - \cdots \tag{1.2}$$

which is antisymmetrised on each index pair, so that

$$F_{\mu_1\mu_2\nu_1\nu_2\rho_1\rho_2\sigma_1\sigma_2} = F_{[\mu_1\mu_2][\nu_1\nu_2][\rho_1\rho_2][\sigma_1\sigma_2]} \tag{1.3}$$

This is the natural generalisation of the linearised Riemann tensor and is represented by the Young tableau . If the free field equation is to come from varying a gauge-invariant action with respect to $A_{\mu\nu\rho\sigma}$, it should be of the form $\mathcal{G}_{\mu\nu\rho\sigma} = 0$ for some gauge-invariant, totally symmetric tensor $\mathcal{G}_{\mu\nu\rho\sigma}$, which is the generalisation of the Ricci tensor. The natural choice is to define

$$G_{\mu_1\nu_1\rho_1\sigma_1} := \eta^{\mu_2\nu_2} \eta^{\rho_2\sigma_2} F_{\mu_1\mu_2\nu_1\nu_2\rho_1\rho_2\sigma_1\sigma_2}\,. \tag{1.4}$$

Then $G_{(\mu_1\nu_1\rho_1\sigma_1)} = 0$ is a covariant local field equation, but is fourth order in derivatives.[1] Here $\eta_{\mu\nu}$ is the background $\mathrm{SO}(D-1,1)$-invariant Minkowski metric, which is used to raise and lower indices. The second order field equation of [11] is then given by

$$\mathcal{G}_{\mu\nu\rho\sigma} := \frac{1}{\Box} G_{(\mu\nu\rho\sigma)} = 0\,. \tag{1.5}$$

In the tranverse traceless gauge (which we will refer to as the 'physical gauge')

$$\partial^\mu A_{\mu\nu\rho\sigma} = 0\,, \qquad \eta^{\mu\nu} A_{\mu\nu\rho\sigma} = 0 \tag{1.6}$$

the equation (1.5) reduces to $\Box A_{\mu\nu\rho\sigma} = 0$, as required (see appendix A for a discusssion of going to transverse traceless gauge).

The generalised Einstein tensor is

$$E_{\mu\nu\rho\sigma} := G_{(\mu\nu\rho\sigma)} - \eta_{(\mu\nu}\, G_{\rho\sigma)\alpha\beta}\, \eta^{\alpha\beta} + \frac{1}{4}\eta_{(\mu\nu}\eta_{\rho\sigma)}\, G_{\alpha\beta\gamma\delta}\, \eta^{\alpha\beta}\eta^{\gamma\delta} \tag{1.7}$$

which satisfies the conservation equation

$$\partial^\mu E_{\mu\nu\rho\sigma} \equiv 0 \tag{1.8}$$

identically. Then the action

$$\mathcal{S}^{[1,1,1,1]}_{(1)} = -\frac{1}{24}\int d^D x\, A^{\mu\nu\rho\sigma}\frac{1}{\Box} E_{\mu\nu\rho\sigma} \tag{1.9}$$

is invariant under the gauge transformation (1.1) and its variation gives the field equation $\frac{1}{\Box}E_{\mu\nu\rho\sigma} = 0$, which is equivalent to (1.5). This is a simple modification of the action $\mathcal{S}^{[1,1,1,1]}$ for a type $[1,1,1,1]$ tensor gauge field given in [9, 10], obtained by inserting the non-local operator $\Box^{-1}$.

The generalisation to totally symmetric tensors $A_{\mu_1\ldots\mu_s}$ of spin-s is straightforward [11]. There is a field strength

$$F_{\mu_1^1\,\mu_2^1\,\mu_1^2\,\mu_2^2\cdots\mu_1^s\,\mu_2^s} = \partial_{\mu_1^1}\partial_{\mu_1^2}\ldots\partial_{\mu_1^s} A_{\mu_2^1\,\mu_2^2\cdots\mu_2^s} - \left(\mu_1^1 \leftrightarrow \mu_2^1\right) - \cdots \tag{1.10}$$

which is antisymmetrised on each of the s index pairs so that

$$F_{\mu_1^1\,\mu_2^1\,\mu_1^2\,\mu_2^2\ldots\mu_1^s\,\mu_2^s} = F_{[\mu_1^1\,\mu_2^1]\,[\mu_1^2\,\mu_2^2]\ldots[\mu_1^s\,\mu_2^s]}\,. \tag{1.11}$$

If s is even, then contracting $F_{\mu_1^1\,\mu_2^1\,\mu_1^2\,\mu_2^2\ldots\mu_1^s\,\mu_2^s}$ over $s/2$ pairs of indices with $s/2$ metric tensors $\eta^{\mu_2^i\,\mu_2^j}$ defines the gauge-invariant tensor $G_{\mu_1\ldots\mu_s}$. A generalised Ricci tensor $G_{(\mu_1\ldots\mu_s)}$ is then defined by total symmetrisation of all s indices and $G_{(\mu_1\ldots\mu_s)} = 0$ is a covariant field equation of order s in derivatives. This is the s derivative field equation of [9, 10] for an even spin-s field. The associated second order field equation of [11] is

$$\mathcal{G}_{\mu_1\ldots\mu_s} := \frac{1}{\Box^r} G_{(\mu_1\ldots\mu_s)} = 0 \tag{1.12}$$

[1] The field equation $G_{\mu_1\nu_1\rho_1\sigma_1} = 0$ (without symmetrisation) is also gauge-invariant but cannot be derived from a gauge-invariant action without introducing extra fields.

JHEP05(2003)019

where $r = \frac{s}{2} - 1$, and reduces to the Fronsdal equations [13] on partial gauge fixing to traceless gauge transformations [11]. These Fronsdal equations then reduce further to $\Box A_{\mu_1 \ldots \mu_s} = 0$ on imposing the physical gauge conditions $\partial^{\mu_1} A_{\mu_1 \ldots \mu_s} = 0$ and $\eta^{\mu_1 \mu_2} \times A_{\mu_1 \mu_2 \ldots \mu_s} = 0$. This equation can be derived from a gauge-invariant non-local action of the form

$$\mathcal{S}^{[1,\ldots,1]}_{(r)} = -\frac{1}{s!} \int d^D x A^{\mu_1 \ldots \mu_s} \frac{1}{\Box^r} E_{\mu_1 \ldots \mu_s} \tag{1.13}$$

where $r = \frac{s}{2} - 1$ and $E_{\mu_1 \ldots \mu_s}$ is the generalised Einstein tensor given by shifting the traces of $G_{\mu_1 \ldots \mu_s}$ and Young symmetrising indices so that $\partial^{\mu_1} E_{\mu_1 \ldots \mu_s} \equiv 0$ identically. The construction of such 'Einstein tensors' is discussed in appendix B.

For odd spins, we define a rank s tensor $G_{\mu_1 \ldots \mu_s}$ by contracting over $\frac{(s+1)}{2}$ pairs of indices of $\partial_{\mu^{s+1}} F_{\mu_1^1 \mu_2^1 \mu_1^2 \mu_2^2 \ldots \mu_1^s \mu_2^s}$. Then the field equation $G_{(\mu_1 \ldots \mu_s)} = 0$ of [9, 10] is of order $s + 1$ in derivatives, and the second order field equation of [11] is as in (1.12), but with $r = \frac{(s+1)}{2} - 1$. This equation follows from a gauge-invariant action of the form (1.13), but with $r = \frac{(s+1)}{2} - 1$. The gauge-invariant tensor $E_{\mu_1 \ldots \mu_s}$ is again defined by shifting the traces of $G_{\mu_1 \ldots \mu_s}$ and symmetrising. In both even and odd spin cases, the physical gauge field equation $\Box A_{\mu_1 \ldots \mu_s} = 0$ follows from the local action

$$\mathcal{S}^{[1,\ldots,1]}_{(r)} = -\frac{1}{s!} \int d^D x \, A^{\mu_1 \ldots \mu_s} \Box \, A_{\mu_1 \ldots \mu_s} \tag{1.14}$$

This generalises to gauge fields in arbitrary representations of the general linear group $GL(D, \mathbb{R})$. Consider for example the Young tableau $\ydiagram{4,1}$ corresponding to a tensor gauge field $A_{\mu_1 \mu_2 \nu \rho \sigma}$ satisfying

$$A_{\mu_1 \mu_2 \nu \rho \sigma} = A_{[\mu_1 \mu_2] \nu \rho \sigma} \,, \qquad A_{[\mu_1 \mu_2 \nu] \rho \sigma} = 0 \,, \qquad A_{\mu_1 \mu_2 \nu \rho \sigma} = A_{\mu_1 \mu_2 (\nu \rho \sigma)} \,. \tag{1.15}$$

We refer to this Young tableau as type $[2, 1, 1, 1]$, where the numbers denote the length of the columns. The field strength is

$$F_{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \rho_1 \rho_2 \sigma_1 \sigma_2} = \partial_{\mu_1} \partial_{\nu_1} \partial_{\rho_1} \partial_{\sigma_1} A_{\mu_2 \mu_3 \nu_2 \rho_2 \sigma_2} - (\nu_1 \leftrightarrow \nu_2) - \cdots \tag{1.16}$$

which is antisymmetrised on each of the four index types so that

$$F_{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \rho_1 \rho_2 \sigma_1 \sigma_2} = F_{[\mu_1 \mu_2 \mu_3][\nu_1 \nu_2][\rho_1 \rho_2][\sigma_1 \sigma_2]} \tag{1.17}$$

and is in a representation that corresponds to the Young tableau $\ydiagram{4,4,1}$ of type $[3, 2, 2, 2]$. Again, one can define a gauge-invariant tensor by

$$G_{\mu_1 \mu_2 \nu_1 \rho_1 \sigma_1} := \eta^{\mu_3 \nu_2} \eta^{\rho_2 \sigma_2} F_{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \rho_1 \rho_2 \sigma_1 \sigma_2} \,. \tag{1.18}$$

This tensor is not irreducible under $GL(D, \mathbb{R})$ but an associated Young tableau can be defined by the Young projection $\mathcal{Y}_{[2,1,1,1]}$ of (1.18) onto the irreducible subspace of type $[2, 1, 1, 1]$ tensors. This projected tensor corresponds to the generalised Ricci tensor and $\mathcal{Y}_{[2,1,1,1]} \circ G_{\mu_1 \mu_2 \nu_1 \rho_1 \sigma_1} = 3 \, \eta^{(\mu_3 \nu_2} \eta^{\rho_2 \sigma_2)} F_{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \rho_1 \rho_2 \sigma_1 \sigma_2} = 0$ is a covariant local fourth order field equation, while $\mathcal{Y}_{[2,1,1,1]} \circ \left(\frac{1}{\Box} G_{\mu_1 \mu_2 \nu_1 \rho_1 \sigma_1} \right) = 0$ is the natural candidate for a second order field equation. Again, an Einstein-type tensor $E_{\mu_1\mu_2\nu_1\rho_1\sigma_1}$ of type $[2,1,1,1]$ can be defined by shifting traces followed by Young projection of $G_{\mu_1\mu_2\nu_1\rho_1\sigma_1}$ (see appendix B), so that the second order field equation above can be derived from the gauge-invariant action

$$\mathcal{S}^{[2,1,1,1]}_{(1)} = -\frac{1}{12}\int d^D x\, A^{\mu_1\mu_2\nu_1\rho_1\sigma_1}\frac{1}{\Box}E_{\mu_1\mu_2\nu_1\rho_1\sigma_1}\,. \tag{1.19}$$

The non-local field equations reduce to $\Box\, A_{\mu_1\mu_2\nu_1\rho_1\sigma_1} = 0$ in the physical gauge

$$\partial^{\mu_1}A_{\mu_1\mu_2\nu_1\rho_1\sigma_1} = 0\,, \qquad \partial^{\nu_1}A_{\mu_1\mu_2\nu_1\rho_1\sigma_1} = 0\,, \qquad \eta^{\mu_1\nu_1}A_{\mu_1\mu_2\nu_1\rho_1\sigma_1} = 0 \tag{1.20}$$

and this field equation follows from the local action

$$\mathcal{S}^{[2,1,1,1]}_{(1)} = -\frac{1}{12}\int d^D x\, A^{\mu_1\mu_2\nu_1\rho_1\sigma_1}\Box\, A_{\mu_1\mu_2\nu_1\rho_1\sigma_1} \tag{1.21}$$

The procedure described above generalises to arbitrary tensor gauge fields A of type $[p_1,\ldots,p_N]$ represented by a Young tableau with N columns, each of length p_i. The field strength F is given by acting with N derivatives on A and projecting onto the representation $[p_1+1,\ldots,p_N+1]$, so that in particular there is an antisymmetrization on the indices in each column. A generalised Ricci tensor of type $[p_1,\ldots,p_N]$ is obtained by first taking $N/2$ traces of F if N is even, or by taking an extra derivative and taking $(N+1)/2$ traces of ∂F if N is odd. By summing over all possible inequivalent ways of taking these traces one obtains the generalised Ricci tensor whose vanishing defines the higher order field equation in [9, 10]. An associated second order field equation is obtained by dividing the higher order equation above by a suitable power r of the D' Alembertian operator. This power is $r = N/2-1$ for even N or $r = (N+1)/2-1$ for odd N. The non-local gauge-invariant action from which these equations derive takes the form $\int d^D x\; A\cdot\frac{1}{\Box^r}E$ in terms of the type $[p_1,\ldots,p_N]$ generalised Einstein tensor E. This non-local action is then replaced by the local form $\int d^D x\; A\cdot\Box\, A$ to derive the field equation $\Box A = 0$ in an appropriate physical gauge.

Clearly, some care is needed in dealing with the index structure in such exotic representations and in making the above prescription precise. For p-form gauge fields, i.e. gauge fields of type $[p]$ with a tableau comprising a single column of length p, differential forms provide the natural formalism to describe the theory. For $N>1$, the generalisation to 'multi-forms' provides the natural formalism to describe the theory. In [9, 10], we presented a theory of multi-forms and applied it to general gauge theories, and in section 2 we review the parts of that which will be used here. Earlier work on the application of differential analysis to higher spin gauge theory appeared in [14, 15] and a similar multi-form construction was used in [16, 17]. While this paper was in preparation, [20] appeared which gave further discussion of the spin-three theory and briefly discussed field equations for general gauge fields.

2. Multi-form gauge theory

We begin by reviewing the multi-form construction of gauge theories with gauge potential transforming in an arbitrary irreducible representation of $GL(D,\mathbb{R})$. Much of the material in this section is given in [9, 10]. For a discussion of Young tableaux, see [18].

2.1 Multi-forms

A *multi-form* of order N is a tensor field T that is an element of the N-fold tensor product of p_i-forms (where $i = 1, \ldots, N$), written

$$X^{p_1,\ldots,p_N} := \Lambda^{p_1} \otimes \ldots \otimes \Lambda^{p_N} \,. \tag{2.1}$$

In general, this will be a reducible representation of $GL(D, \mathbb{R})$. The components of T are written $T_{\mu^1_1 \ldots \mu^1_{p_1} \ldots \mu^i_1 \ldots \mu^i_{p_i} \ldots \mu^N_{p_N}}$ and are taken to be totally antisymmetric in each set of $\{\mu^i\}$ indices, so that

$$T_{\mu^1_1 \ldots \mu^1_{p_1} \ldots \mu^i_1 \ldots \mu^i_{p_i} \ldots \mu^N_1 \ldots \mu^N_{p_N}} = T_{[\mu^1_1 \ldots \mu^1_{p_1}] \ldots [\mu^i_1 \ldots \mu^i_{p_i}] \ldots [\mu^N_1 \ldots \mu^N_{p_N}]} \,. \tag{2.2}$$

The generalisation of the operations on ordinary differential forms to multi-forms of order N over $\mathbb{R}^D$ is as follows.

The $\odot$-product is the natural generalisation of wedge product to multi-forms and is given by the map

$$\odot : X^{p_1,\ldots,p_N} \times X^{p'_1,\ldots,p'_N} \to X^{p_1+p'_1,\ldots,p_N+p'_N} \tag{2.3}$$

defined by the N-fold wedge product on the individual form subspaces.

There are N inequivalent exterior derivatives

$$d^{(i)} : X^{p_1,\ldots,p_i,\ldots p_N} \to X^{p_1,\ldots,p_i+1,\ldots,p_N} \tag{2.4}$$

which are individually defined as the exterior derivatives acting on the Λ^{p_i} form subspaces. This definition implies ${d^{(i)}}^2 = 0$ (with no sum over i) and that $d^{(i)}$ commutes with $d^{(j)}$ for any i, j. Since the multi-form space $X^{p_1,\ldots,p_N}$ is isomorphic to the multi-form space $X^{p_1,\ldots,p_N,0}$ of one order higher, one can introduce a further derivative operator on $X^{p_1,\ldots,p_N}$, defined by

$$\partial := d^{(N+1)} : X^{p_1,\ldots,p_N} \to X^{p_1,\ldots,p_N,1} \,. \tag{2.5}$$

This follows by taking the partial derivative of a multi-form $T \in X^{p_1,\ldots,p_N}$ to define an element in $X^{p_1,\ldots,p_N,1}$ with components $\partial_{\mu^{N+1}_1} T_{\mu^1_1 \ldots \mu^1_{p_1} \ldots \mu^i_1 \ldots \mu^i_{p_i} \ldots \mu^N_{p_N}}$. One can also define the total derivative

$$\mathcal{D} := \sum_{i=1}^{N} d^{(i)} \,, \qquad \mathcal{D} : X^{p_1,\ldots,p_i,\ldots p_N} \to \sum_{i=1}^{N} \oplus X^{p_1,\ldots,p_i+1,\ldots,p_N} \tag{2.6}$$

which satisfies $\mathcal{D}^{N+1} = 0$.

For representations of $\mathrm{SO}(D-1,1) \subset GL(D, \mathbb{R})$ there are N inequivalent Hodge dual operations

$$*^{(i)} : X^{p_1,\ldots,p_i,\ldots p_N} \to X^{p_1,\ldots,D-p_i,\ldots p_N} \tag{2.7}$$

which are defined to act as the Hodge duals on the individual Λ^{p_i} form subspaces. This implies that ${*^{(i)}}^2 = (-1)^{1+p_i(D-p_i)}$ (with no sum over i) and that $*^{(i)}$ commutes with $*^{(j)}$ for any i, j.

This also allows one to define N inequivalent 'adjoint' exterior derivatives

$${d^{\dagger}}^{(i)} := (-1)^{1+D(p_i+1)} *^{(i)} d^{(i)} *^{(i)} : X^{p_1,\ldots,p_i,\ldots p_N} \to X^{p_1,\ldots,p_i-1,\ldots,p_N} \,. \tag{2.8}$$

This implies ${d^{\dagger}}^{(i)^2} = 0$ (with no sum over i) and any two ${d^{\dagger}}^{(i)}$ commute. One can then define the laplacian operator

$$\Delta := d^{(i)}{d^{\dagger}}^{(i)} + {d^{\dagger}}^{(i)}d^{(i)} : X^{p_1,\dots,p_i,\dots p_N} \to X^{p_1,\dots,p_i,\dots,p_N} \tag{2.9}$$

with no sum over i. The action of the laplacian operator on multi-form $T \in X^{p_1,\dots,p_N}$ is independent of which $i = 1,\dots,N$ is chosen for Δ. This can be seen in component form since $(\Delta T)_{\mu^1_1\dots\mu^1_{p_1}\dots\mu^i_1\dots\mu^i_{p_i}\dots\mu^N_{p_N}} = \Box T_{\mu^1_1\dots\mu^1_{p_1}\dots\mu^i_1\dots\mu^i_{p_i}\dots\mu^N_{p_N}}$, where $\Box := \partial_\mu\partial^\mu$ is the D' Alembertian operator on $\mathbb{R}^{D-1,1}$.

There exist $N(N-1)/2$ inequivalent trace operations

$$\tau^{(ij)} : X^{p_1,\dots,p_i,\dots,p_j,\dots p_N} \to X^{p_1,\dots,p_i-1,\dots,p_j-1,\dots,p_N} \tag{2.10}$$

defined as the single trace between the Λ^{p_i} and Λ^{p_j} form subspaces using the Minkowski metric $\eta^{\mu^i_1\mu^j_1}$. This allows one to define two inequivalent 'dual-trace' operations

$$\sigma^{(ij)} := (-1)^{1+D(p_i+1)} *^{(i)} \tau^{(ij)} *^{(i)} : X^{p_1,\dots,p_i,\dots,p_j,\dots p_N} \to X^{p_1,\dots,p_i+1,\dots,p_j-1,\dots,p_N} \tag{2.11}$$

and

$$\tilde{\sigma}^{(ij)} := (-1)^{1+D(p_j+1)} *^{(j)} \tau^{(ij)} *^{(j)} : X^{p_1,\dots,p_i,\dots,p_j,\dots p_N} \to X^{p_1,\dots,p_i-1,\dots,p_j+1,\dots,p_N} \tag{2.12}$$

associated with a given $\tau^{(ij)}$ (with no sum over i or j). Notice that $\tilde{\sigma}^{(ij)} = \sigma^{(ji)}$ since $\tau^{(ij)} = \tau^{(ji)}$. This implies that the components $(\sigma^{(ij)}T)_{\mu^1_1\dots[\mu^i_1\dots\mu^i_{p_i}\mu^j_1]\dots[\mu^j_2\dots\mu^j_{p_j}]\dots\mu^N_{p_N}}$ are equal to $(-1)^{p_i+1}(p_i+1)\,T_{\mu^1_1\dots[\mu^i_{p_1}\dots\mu^i_{p_i}|\dots|\mu^j_1]\mu^j_2\dots\mu^j_{p_j}\dots\mu^N_{p_N}}$.

There are $N(N-1)/2$ inequivalent involutions

$$t^{(ij)} : X^{p_1,\dots,p_i,\dots,p_j,\dots p_N} \to X^{p_1,\dots,p_j,\dots,p_i,\dots,p_N} \tag{2.13}$$

defined by exchange of the Λ^{p_i} and Λ^{p_j} form subspaces in the tensor product space. The components $(t^{(ij)}T)_{\mu^1_1\dots\mu^j_1\dots\mu^j_{p_j}\dots\mu^i_1\dots\mu^i_{p_i}\dots\mu^N_{p_N}}$ are proportional to

$$T_{\mu^1_1\dots[\mu^j_1\dots\mu^j_{p_j}][\mu^i_1\dots\mu^i_{p_i-p_j}|\dots|\mu^i_{p_i-p_j+1}\dots\mu^i_{p_i}]\dots\mu^N_{p_N}}$$

(assuming $p_i \geq p_j$).

There are also $N(N-1)/2$ inequivalent product operations

$$\eta^{(ij)} : X^{p_1,\dots,p_i,\dots,p_j,\dots p_N} \to X^{p_1,\dots,p_i+1,\dots,p_j+1,\dots,p_N} \tag{2.14}$$

defined as the $\odot$-product with the SO$(D-1,1)$ metric η (understood as an order N multi-form with all columns of zero length except $p_i = p_j = 1$, corresponding to a $[1,1]$ bi-form in the $\Lambda^{p_i} \otimes \Lambda^{p_j}$ subspace), such that $\eta^{(ij)}T \equiv \eta \odot T$ for any $T \in X^{p_1,\dots,p_N}$. The components

$$(\eta^{(ij)}T)_{\mu^1_1\dots\mu^i_1\dots\mu^i_{p_i+1}\dots\mu^j_1\dots\mu^j_{p_j+1}\dots\mu^N_{p_N}}$$

are equal to

$$(p_i+1)(p_j+1)\,\eta_{\mu^i_1\mu^j_1}T_{\mu^1_1\dots\mu^i_2\dots\mu^i_{p_i+1}\dots\mu^j_2\dots\mu^j_{p_j+1}\dots\mu^N_{p_N}}$$

(with implicit antisymmetrisation on the (p_i+1) μ^i and (p_j+1) μ^j indices seperately).

JHEP05(2003)019

2.2 Irreducible representations and Young tableaux

The space of multi-forms $X^{p_1,\ldots,p_N}$ is in general a reducible representation of $GL(D,\mathbb{R})$. Each irreducible representation of $GL(D,\mathbb{R})$ is associated with a Young tableau. Consider the representation associated with a Young tableau with N columns and with p_i cells in the ith column (it is assumed $p_i \geq p_{i+1}$); we denote this representation as $[p_1,\ldots,p_N]$. A tensor A in this representation is a multi-form $A \in X^{p_1,\ldots,p_N}$ satisfying

$$\sigma^{(ij)}A = 0 \tag{2.15}$$

for any $j > i$ and also satisfying $t^{(ij)}A = A$ if the ith and jth columns are of equal length, $p_i = p_j$ [18]. The projector from $X^{p_1,\ldots,p_N}$ onto this irreducible tensor representation $X^{[p_1,\ldots,p_N]}$ is the Young symmetriser $\mathcal{Y}_{[p_1,\ldots,p_N]}$.

For example, consider a rank s multi-form in $X^{1,1,\ldots,1}$ (with $N = s$ and all $p_i = 1$). This is the space of all rank s tensors $T_{\mu_1\mu_2\ldots\mu_s}$, with no index symmetry properties and so is a reducible representation of $GL(D,\mathbb{R})$. The projector $\mathcal{Y}_{[s,0,\ldots,0]}$ takes this to the space of totally antisymmetric tensors $T_{\mu_1\mu_2\ldots\mu_s} = T_{[\mu_1\mu_2\ldots\mu_s]}$, while $\mathcal{Y}_{[1,1,\ldots,1]}$ takes this to the space of totally symmetric tensors $T_{\mu_1\mu_2\ldots\mu_s} = T_{(\mu_1\mu_2\ldots\mu_s)}$. The full set of irreducible representations are obtained by acting on $X^{1,1,\ldots,1}$ with all projectors $\mathcal{Y}_{[p_1,\ldots,p_s]}$ with $p_i \geq p_{i+1}$ satisfying $\sum_{i=1}^{s} p_i = s$.

2.3 Multi-form gauge theory

Consider a gauge potential A that is a tensor in the $[p_1,\ldots,p_N]$ irreducible representation of $GL(D,\mathbb{R})$ whose components have the index symmetry of an N-column Young tableau with p_i cells in the ith column (with $p_i \geq p_{i+1}$). The natural gauge transformation for this object is given by [10]

$$\delta A = \mathcal{Y}_{[p_1,\ldots,p_N]} \circ \left(\sum_{i=1}^{N} d^{(i)} \alpha_{(i)}^{p_1,\ldots,p_i-1,\ldots,p_N} \right) \tag{2.16}$$

for any gauge parameters $\alpha_{(i)}^{p_1,\ldots,p_i-1,\ldots,p_N} \in X^{p_1,\ldots,p_i-1,\ldots,p_N}$.

The associated field strength F is a type $[p_1+1,\ldots,p_N+1]$ tensor given by

$$F = \left(\prod_{i=1}^{N} d^{(i)} \right) A = \frac{1}{N!} \mathcal{D}^N A \tag{2.17}$$

which is invariant under (2.16). The first expression is unambiguous since all $d^{(i)}$ commute.

From the generalised Poincaré lemma in [14, 15], [16] it follows that any type $[p_1 + 1,\ldots,p_N+1]$ tensor F satisfying $d^{(i)}F = 0$ for all i can be written as in (2.17) for some type $[p_1,\ldots,p_N]$ potential A. The field strength F satisfies second Bianchi identities

$$d^{(i)}F = 0 \tag{2.18}$$

and the first Bianchi identities

$$\sigma^{(ij)}F = 0 \tag{2.19}$$

for any $j > i$.

Considering the irreducible representations of $GL(D,\mathbb{R})$ above to be reducible representations of the $\mathrm{SO}(D-1,1)$ Lorentz subgroup allows the construction of a gauge-invariant action functional from which physical equations of motion can be obtained.

For N odd, the natural field equation for a general type $[p_1,\ldots,p_N]$ gauge potential A is given by

$$\sum_{I\in S_N} \tau^{(i_1 i_2)} \ldots \tau^{(i_N\, N+1)} \partial F = 0 \tag{2.20}$$

where the sum is on the labels $I = (i_1 \ldots i_N)$ whose values vary over all permutations of the set $(1\ldots N)$. The $(N+1)$th label is not included in the sum. The fact that the Young projection $\mathcal{Y}_{[p_1,\ldots,p_N,0]}$ onto the irreducible $[p_1,\ldots,p_N,0]$ tensor subspace is not required in this expression is shown in appendix B. For N even, the field equation for a type $[p_1,\ldots,p_N]$ gauge potential A is given by

$$\sum_{I\in S_N} \tau^{(i_1 i_2)} \ldots \tau^{(i_{N-1} i_N)} F = 0 \tag{2.21}$$

where the sum here is on all the labels $I = (i_1 \ldots i_N)$ whose values vary over all permutations of the set $(1\ldots N)$. The Young projection $\mathcal{Y}_{[p_1,\ldots,p_N]}$ is again unnecessary.

If these field equations can be derived from a gauge-invariant action, it must be of the form

$$\mathcal{S}^{[p_1,\ldots,p_N]} \;=\; -\left(\prod_{i=1}^{N} \frac{1}{p_i!}\right) \int d^D x\; A^{\mu^1_1\ldots\mu^1_{p_1}\ldots\mu^i_1\ldots\mu^i_{p_i}\ldots\mu^N_{p_N}} E_{\mu^1_1\ldots\mu^1_{p_1}\ldots\mu^i_1\ldots\mu^i_{p_i}\ldots\mu^N_{p_N}} \tag{2.22}$$

in terms of the type $[p_1,\ldots,p_N]$ gauge potential A and some gauge-invariant field equation tensor E involving N partial derivatives on A for even N (or $N+1$ derivatives for odd N). Gauge invariance of (2.22) requires that E should satisfy the N conservation conditions $\partial^{\mu^i_1} E_{\mu^1_1\ldots\mu^1_{p_1}\ldots\mu^i_1\ldots\mu^i_{p_i}\ldots\mu^N_{p_N}} \equiv 0$ identically for $i = 1,\ldots,N$. For N even, the leading term in E involves $N/2$ traces of the field strength F of A and is given by the sum over all permutations of N labels of the term $F_{\mu^1_1\ldots\mu^1_{p_1+1}\ldots\mu^N_{p_N+1}}\; \eta^{\mu^1_1\,\mu^2_1} \ldots \eta^{\mu^{N-1}_1\,\mu^N_1}$. The correction terms then consist of further traces (appropriately symmetrised) with coefficients fixed by overall conservation of E [10], so that the field equation $E=0$ is a linear combination of the field equation given above and its multiple trace parts. For N odd one can consider the potential to be a type $[p_1,\ldots,p_N,0]$ tensor of even order $N+1$ whose field strength ∂F is a type $[p_1+1,\ldots,p_N+1,1]$ tensor. The construction of E is then the same as for the even N case. The explicit form of E is discussed in appendix B, where it is constructed explicitly for bi-forms and totally symmetric tensors, and a general form is conjectured.

For general N the local gauge-invariant field equations above are Nth order in derivatives for even N and of order $N+1$ for odd N. Consequently they are higher derivative equations of motion for higher spin fields with $N > 2$ involving more than two partial derivatives of the gauge field. In the next section we give a method for obtaining second order field equations for such higher spin tensor fields with arbitrary N. This construction was described in [11] for the case of totally symmetric spin-s gauge fields. In our notation such totally symmetric spin-s fields are tensors of type $[1,\ldots,1]$ with $N = s$ entries.

JHEP05(2003)019

3. Second order field equations

The local gauge-invariant action (2.22) for a general type $[p_1, \ldots, p_N]$ gauge potential A can be modified by the insertion of a negative power of the D' Alembertian scalar operator $\Box$. The resulting non-local action is given by

$$\mathcal{S}_{(r)}^{[p_1,\ldots,p_N]} = -\left(\prod_{i=1}^{N} \frac{1}{p_i!}\right) \int d^D x \, A^{\mu_1^1 \ldots \mu_{p_1}^1 \ldots \mu_1^i \ldots \mu_{p_i}^i \ldots \mu_{p_N}^N} \frac{1}{\Box^r} E_{\mu_1^1 \ldots \mu_{p_1}^1 \ldots \mu_1^i \ldots \mu_{p_i}^i \ldots \mu_{p_N}^N} \tag{3.1}$$

and is gauge-invariant for any power r. Formally, this gives field equations of order $N-2r$ for N even or $N+1-2r$ for N odd, so that choosing $r = \frac{N}{2}-1$ for N even and $r = \frac{(N+1)}{2}-1$ for N odd gives second order field equations. For any r, these field equations are covariant and gauge-invariant, but are non-local in general. We will show that the second order field equations become local in physical gauge, and it is to be expected that the non-localities could instead be eliminated by the introduction of auxiliary fields, as shown for the spin-three case in [11]. For the case in which r is chosen to make the field equation zero'th order, the field equations imply the fields are pure gauge.

Choosing $r = \frac{N}{2} - 1$ for N even and $r = \frac{(N+1)}{2} - 1$ for N odd, gives gauge-invariant field equations derived from (3.1) that are of second order and are given by the non-local expressions

$$\mathcal{G}^{(0)} := \sum_{I \in S_N} \tau^{(i_1 i_2)} \ldots \tau^{(i_{N-1} i_N)} \frac{1}{\Box^{\frac{N}{2}-1}} F = 0 \tag{3.2}$$

$$\mathcal{G}^{(1)} := \sum_{I \in S_N} \tau^{(i_1 i_2)} \ldots \tau^{(i_N \, N+1)} \frac{1}{\Box^{\frac{(N+1)}{2}-1}} \partial F = 0 \tag{3.3}$$

for N even and odd respectively. The Young projections $\mathcal{Y}_{[p_1,\ldots,p_N]}$ and $\mathcal{Y}_{[p_1,\ldots,p_N,0]}$ in (3.2) and (3.3) are not necessary, following the theorem in appendix B. These equations correspond to those proposed in [11] for the case of a spin-s gauge field with $N = s$ and all $p_i = 1$. The non-local action (3.1) is then replaced by the local action

$$\mathcal{S}_{(r)}^{[p_1,\ldots,p_N]} = -\left(\prod_{i=1}^{N} \frac{1}{p_i!}\right) \int d^D x \, A^{\mu_1^1 \ldots \mu_{p_1}^1 \ldots \mu_1^i \ldots \mu_{p_i}^i \ldots \mu_{p_N}^N} \Box \, A_{\mu_1^1 \ldots \mu_{p_1}^1 \ldots \mu_1^i \ldots \mu_{p_i}^i \ldots \mu_{p_N}^N} \tag{3.4}$$

in the physical gauge

$$d^{\dagger (i)} A = 0 \,, \qquad \tau^{(ij)} A = 0 \tag{3.5}$$

for any $i, j = 1, \ldots, N$.

The second order, non-local field equations above are not unique. Define $F^{(m)} := \partial^m F$ to be the order $N+m$ tensor associated with the canonical field strength tensor F of order N obtained by acting on F with m partial derivatives. Then write

$$\mathcal{G}^{(2n)} := \sum_{I \in S_{N+2n}} \tau^{(i_1 i_2)} \ldots \tau^{(i_{N-1+2n} \, i_{N+2n})} \frac{1}{\Box^{\frac{N}{2}-1+n}} F^{(2n)} \tag{3.6}$$

$$\mathcal{G}^{(2n+1)} := \sum_{I \in S_{N+2n}} \tau^{(i_1 i_2)} \ldots \tau^{(i_{N+2n} \, N+1+2n)} \frac{1}{\Box^{\frac{(N+1)}{2}-1+n}} F^{(2n+1)} \tag{3.7}$$

for the case of N even and odd respectively. The Young projections $\mathcal{Y}_{[p_1,\ldots,p_N,0,\ldots,0]}$ (with $2n$ zeros) and $\mathcal{Y}_{[p_1,\ldots,p_N,0,0,\ldots,0]}$ (with $2n+1$ zeros) in (3.6) and (3.7) respectively, are not necessary. It is clear by construction that (3.6) and (3.7) are related to the original field equation tensors in (3.2) and (3.3) and the equation $\mathcal{G}^{(m)}=0$ also reduces to $\Box A=0$ if one imposes the physical gauge conditions (3.5). The field equations given by the vanishing of (3.6) and (3.7) are more restrictive than (3.2) and (3.3) in the sense that the gauge-invariant tensors in (3.6) and (3.7) vanish as a consequence of (3.2) and (3.3) though the converse statement is not true.[2] One of these associated field equations is noted in [11] for the case of a spin-3 field. In our framework this example corresponds to the case in which $N=3$, $p_1=p_2=p_3=1$ and $n=1$. As noted in [11], this second order field equation $\mathcal{G}^{(3)}=0$ is simply related to a linear combination of the equation $\mathcal{G}^{(1)}=0$ and its trace. In general, one can consider field equations given by linear combinations of these tensors, $\sum_n a_n\,\mathcal{G}^{(2n)}=0$ or $\sum_n a_n\,\mathcal{G}^{(2n+1)}=0$ for some coefficients a_n, but these will generally be more restrictive than (3.2) and (3.3).

4. Connections

The linearised Riemann tensor can be written as the (appropriately symmetrised) single derivative of a first order linearised connection, such that $R_{\mu\nu\rho\sigma}=4\partial_{[\mu}\Gamma_{\nu][\rho\sigma]}$ where $\Gamma_{\nu\rho\sigma}=\frac{1}{2}(\partial_\nu h_{\rho\sigma}+\partial_\rho h_{\nu\sigma}-\partial_\sigma h_{\nu\rho})$. More generally, [11, 19], the field strength of a general spin-s gauge field can be written as a derivative of a rank $2s-1$ linear connection involving $s-1$ derivatives of the spin-s gauge field. We show that such a linear connection structure arises for general tensor gauge theories.

The type $[p_1+1,\ldots,p_N+1]$ tensor field strength associated with an arbitrary type $[p_1,\ldots,p_N]$ gauge potential A satisfies $d^{(i)}F=0$ for all i. For any given i, this implies that F is $d^{(i)}$-exact, so that

$$F=d^{(i)}\tilde{\Gamma}_{(i)} \tag{4.1}$$

(with no sum over i) where $\tilde{\Gamma}_{(i)}\in X^{p_1+1,\ldots,p_i,\ldots,p_N+1}$ is defined by

$$\tilde{\Gamma}_{(i)}:=\left(\prod_{j\neq i}d^{(j)}\right)A\,. \tag{4.2}$$

Notice that $\tilde{\Gamma}_{(i)}$ is a multi-form involving $N-1$ derivatives of A but is not $GL(D,\mathbb{R})$-irreducible in general. Under the gauge transformation (2.16), $\tilde{\Gamma}_{(i)}$ transforms as

$$\delta\tilde{\Gamma}_{(i)}=\left(\prod_{j=1}^{N}d^{(j)}\right)\alpha_{(i)}^{p_1,\ldots,p_i-1,\ldots,p_N} \tag{4.3}$$

[2] A simple example to illustrate this fact is for linearised gravity where $\mathcal{G}^{(0)}_{\mu\nu}=R_{\mu\nu}$ is the Ricci tensor and $\mathcal{G}^{(2)}_{\mu\nu}=R_{\mu\nu}-2\frac{1}{\Box}\partial^\rho\partial^\sigma R_{\rho\mu\sigma\nu}$ where $R_{\mu\nu\rho\sigma}$ is the full linearised Riemann tensor. It is clear that the non-trivial Einstein equation $R_{\mu\nu}=0$ in $D\geq4$ implies the secondary field equation $\partial^\mu R_{\mu\nu\rho\sigma}=0$ (by tracing the second Bianchi identity $\partial_{[\alpha}R_{\mu\nu]\rho\sigma}=0$) so that $\mathcal{G}^{(0)}=0$ implies $\mathcal{G}^{(2)}=0$ but not vice versa. This structure follows in the general theory where one expands a given field equation of level m (i.e. $\mathcal{G}^{(m)}=0$) and finds only lower field equations of levels $<m$ and their various 'secondary' field equations derived using these lower level equations and the associated Bianchi identities.

and is invariant under the transformations with parameter $\alpha_{(j)}$ for any $j \neq i$, so that F is gauge-invariant. All N $\tilde{\Gamma}_{(i)}$ are inequivalent if no two column lengths p_i are equal. For any two columns i and j of equal length (with $p_i = p_j$) then $\tilde{\Gamma}_{(i)}$ and $\tilde{\Gamma}_{(j)}$ are equivalent under transposition, in the sense that $\tilde{\Gamma}_{(i)} = t^{(ij)}\tilde{\Gamma}_{(j)}$ (with no sum over j). This explains why only a single linear connection is realised in [11, 19] for a spin-s gauge field with all $p_i = 1$.

Note that for gravity, the linearised Christoffel connection Γ is related to the bi-form connection $\tilde{\Gamma}$ defined in this way by $2\Gamma_{\mu[\nu\rho]} = -2h_{\mu[\nu,\rho]} \equiv \tilde{\Gamma}_{\mu\nu\rho} \in X^{1,2}$ and is distinguished by its transformation property $\delta\Gamma_{\mu\nu\rho} = \partial_\mu\partial_\nu\xi_\rho$ under $\delta h_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)}$. For totally symmetric tensors, generalisations of the linearised Christoffel connection Γ were proposed in [19], and these are related to linear combinations of the $\tilde{\Gamma}$, but for the general case, the multi-forms $\tilde{\Gamma}_{(i)}$ seem to be more natural in the linear theory.

For a totally symmetric spin-s field, there is in fact a hierarchy of connections [19], and the same is true for general $[p_1, \ldots, p_N]$ tensor gauge fields A with gauge-invariant field strength $F = d^{(1)} \ldots d^{(N)}A$. In addition to the N multi-forms $\tilde{\Gamma}_{(i)}$ of order $N-1$ in derivatives defined above, one can define $N!/(N-k)!k!$ multi-forms of order $N-k$ in derivatives as

$$\tilde{\Gamma}_{(i_1 \ldots i_k)} := \left(\prod_{i \notin (i_1, \ldots, i_k)} d^{(i)} \right) A \in X^{p_1+1, \ldots, p_{i_1}, \ldots, p_{i_k}, \ldots, p_N+1} \tag{4.4}$$

by pulling off k different exterior derivatives from the definition of F in all possible inequivalent ways. Consequently, at the top of the hierarchy there is one $\tilde{\Gamma} = F$ and at the bottom there is also one inequivalent $\tilde{\Gamma}_{(i_1 \ldots i_N)} = A$. It is always possible to write F in terms of any one of these multiforms since

$$F = d^{(i_1)} \ldots d^{(i_k)} \tilde{\Gamma}_{(i_1 \ldots i_k)} \,. \tag{4.5}$$

Therefore, by construction, each multi-form $\tilde{\Gamma}_{(i_1 \ldots i_k)}$ transforms in such a way that F is invariant under gauge transformation (2.16).

5. First order fermionic field equations

It is straightforward to generalise the analysis of a bosonic tensor gauge field in the representation $[p_1, \ldots, p_N]$ to the case of a fermionic spinor-valued tensor gauge field in the representation $[p_1, \ldots, p_N]_S := [p_1, \ldots, p_N] \otimes \mathsf{S}$ of (the cover of) the Lorentz group, given by the tensor product of the tensor representation with the Dirac spinor representation S (with no constraints on traces or gamma-traces).

For a Dirac spinor $\psi \in X^{[0]_S}$, the Dirac equation $\partial\!\!\!/ \psi = 0$ implies the Klein-Gordon equation $\Box \psi = 0$ (where $\partial\!\!\!/ := \gamma^\mu \partial_\mu$ and $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$) while the Dirac equation can formally be obtained by acting on the Klein-Gordon equation $\Box \psi = 0$ with the non-local operator $\frac{1}{\Box}\partial\!\!\!/$. Similarly for a Dirac spinor-valued vector (gravitino) $\psi \in X^{[1]_S}$, the Rarita-Schwinger equation $\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho = 0$ (where $\gamma^{\mu\nu\rho} := \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$) implies the Maxwell equation $\partial^\mu\partial_{[\mu}\psi_{\nu]} = 0$, and conversely acting on the Maxwell equation with $\frac{1}{\Box}\partial\!\!\!/$ gives an equation equivalent to the Rarita-Schwinger equation [11].

In [13] first order field equations were given for general spinor-valued totally symmetric rank s tensor gauge fields (referred to as spin-$(s+1/2)$ fields) in the $[1,1,\ldots,1]_S$ representation, which are invariant under gauge transformations with constrained parameters. In [11], a non-local form of these equations was found which is invariant under gauge transformations with unconstrained parameters. Consider the case of a spin-5/2 tensor field $\psi \in X^{[1,1]_S}$ whose first order field equation in [13] is

$$\partial\!\!\!/ \psi_{\mu\nu} - 2\partial_{(\mu}\psi\!\!\!/_{\nu)} = 0 \tag{5.1}$$

where $\psi\!\!\!/_\nu := \gamma^\mu \psi_{\mu\nu}$. (5.1) is only gauge-invariant under

$$\delta\psi_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)} \tag{5.2}$$

if the spin-3/2 parameter satisfies the constraint $\gamma^\mu \xi_\mu = 0$. A non-local fully gauge-invariant field equation [11] is obtained by taking the linear combination

$$\partial\!\!\!/ \psi_{\mu\nu} - 2\partial_{(\mu}\psi\!\!\!/_{\nu)} - \frac{\partial_\mu \partial_\nu}{\Box}\left(\partial\!\!\!/ \psi^\rho{}_\rho - 2\partial^\rho \psi\!\!\!/_\rho\right) = 0 \tag{5.3}$$

of (5.1) with its trace. This is invariant under (5.2) with unconstrained parameter. Acting on (5.3) with $\partial\!\!\!/$ one obtains the second order linearised Einstein equation

$$\eta^{\mu\rho}\partial_{[\mu}\psi_{\nu][\rho,\sigma]} = 0 \tag{5.4}$$

which is gauge-invariant and local. Conversely, one obtains (5.3) from (5.4) by acting with $\frac{1}{\Box}\partial\!\!\!/$ on the latter. The generalisation to arbitrary spinor-valued spin-s fermionic fields is then straightforward [11]; one obtains fully gauge-invariant field equations by taking non-local linear combinations of the field equation

$$\partial\!\!\!/ \psi_{\mu_1\ldots\mu_s} - s\partial_{(\mu_1}\psi\!\!\!/_{\mu_2\ldots\mu_s)} = 0 \tag{5.5}$$

from [13], where $\psi\!\!\!/_{\mu_2\ldots\mu_s} := \gamma^{\mu_1}\psi_{\mu_1\ldots\mu_s}$. A second order field equation is obtained by acting on the first order gauge-invariant field equation with $\partial\!\!\!/$, and this second order equation for a spinor-valued spin-$(s+1/2)$ fermionic field is that discussed in previous sections for a spin-s bosonic field but with A replaced with ψ. The first order field equation is regained by acting on this second order equation with $\frac{1}{\Box}\partial\!\!\!/$.

5.1 First order equations for general $[p_1,\ldots,p_N]_S$ tensors

This generalises to general spinor-valued tensor fields. The operations on multi-forms extend trivially to spinor-valued multi-forms. The local gauge-invariant action for a fermionic field $\psi \in X^{[p_1,\ldots,p_N]_S}$ is

$$\mathcal{S}^{[p_1,\ldots,p_N]_S} = -\left(\prod_{i=1}^{N}\frac{1}{p_i!}\right)\int d^Dx\, \bar{\psi}^{\,\mu_1^1\ldots\mu_{p_1}^1\ldots\mu_1^i\ldots\mu_{p_i}^i\ldots\mu_{p_N}^N} E_{\mu_1^1\ldots\mu_{p_1}^1\ldots\mu_1^i\ldots\mu_{p_i}^i\ldots\mu_{p_N}^N}(\psi) \tag{5.6}$$

where $\bar{\psi}$ denotes the Dirac conjugate of ψ. The gauge-invariant type $[p_1,\ldots,p_N]_S$ fermionic field equation tensor $E(\psi)$ involves N partial derivatives on ψ for even N (or $N+1$

derivatives for odd N) and E is identical, as an operator, to that given earlier in terms of derivatives of A. In particular, E again satisfies the N conservation conditions $d^{\dagger(i)}E \equiv 0$ identically for $i = 1, \dots, N$. For N even, the fermionic field equation derived from (5.6), for a general type $[p_1, \dots, p_N]_S$ fermion ψ is given by

$$\sum_{I \in S_N} \tau^{(i_1 i_2)} \dots \tau^{(i_{N-1} i_N)} F(\psi) = 0 \tag{5.7}$$

whilst for N odd, the derived fermionic field equation is given by

$$\sum_{I \in S_N} \tau^{(i_1 i_2)} \dots \tau^{(i_N\, N+1)} \partial\!\!\!/ F(\psi) = 0 \tag{5.8}$$

where $F(\psi) = d^{(1)} \dots d^{(N)} \psi$ is the fermionic type $[p_1 + 1, \dots, p_N + 1]_S$ tensor field strength for ψ.

Non-local first order field equations can be obtained from these local higher derivative equations by acting with $\frac{1}{\Box^r}\partial\!\!\!/$ for suitable r. The non-local gauge-invariant action is

$$\mathcal{S}^{[p_1,\dots,p_N]_S}_{(r+1/2)} = -\left(\prod_{i=1}^{N} \frac{1}{p_i!}\right) \int d^D x\, \bar{\psi}^{\,\mu^1_1 \dots \mu^1_{p_1} \dots \mu^i_1 \dots \mu^i_{p_i} \dots \mu^N_{p_N}} \frac{\partial\!\!\!/}{\Box^{r+1}} E_{\mu^1_1 \dots \mu^1_{p_1} \dots \mu^i_1 \dots \mu^i_{p_i} \dots \mu^N_{p_N}}(\psi) \tag{5.9}$$

where, as in (3.1), the power r is chosen to be $\frac{N}{2} - 1$ for N even and $\frac{(N+1)}{2} - 1$ for N odd so that the derived field equations are of first order. The gauge-invariant field equations are given by

$$\mathcal{G}^{(0)}_S := \sum_{I \in S_N} \tau^{(i_1 i_2)} \dots \tau^{(i_{N-1} i_N)} \frac{\partial\!\!\!/}{\Box^{\frac{N}{2}}} F(\psi) = 0 \tag{5.10}$$

$$\mathcal{G}^{(1)}_S := \sum_{I \in S_N} \tau^{(i_1 i_2)} \dots \tau^{(i_N\, N+1)} \frac{\partial\!\!\!/}{\Box^{\frac{(N+1)}{2}}} \partial\!\!\!/ F(\psi) = 0 \tag{5.11}$$

for N even and odd respectively. For spinor-valued spin-s fields (with $N = s$ and all $p_i = 1$) these gauge-invariant fermionic field equations correspond to those proposed in [11].

As in the bosonic case, one can construct associated first order non-local field expressions

$$\mathcal{G}^{(2n)}_S := \sum_{I \in S_{N+2n}} \tau^{(i_1 i_2)} \dots \tau^{(i_{N-1+2n}\, i_{N+2n})} \frac{\partial\!\!\!/}{\Box^{\frac{N}{2}+n}} F^{(2n)}(\psi) \tag{5.12}$$

$$\mathcal{G}^{(2n+1)}_S := \sum_{I \in S_{N+2n}} \tau^{(i_1 i_2)} \dots \tau^{(i_{N+2n}\, N+1+2n)} \frac{\partial\!\!\!/}{\Box^{\frac{(N+1)}{2}+n}} F^{(2n+1)}(\psi) \tag{5.13}$$

for the case of N even and odd respectively. The field equation (5.10) implies the vanishing of (5.12) for all n while (5.11) implies that (5.13) are zero. In general, one can also consider general gauge-invariant field equations which are the linear combinations $\sum_n a_n\, \mathcal{G}^{(2n)}_S = 0$ or $\sum_n a_n\, \mathcal{G}^{(2n+1)}_S = 0$ for some coeffiecients a_n.

The non-local action is replaced by the local Dirac form

$$\mathcal{S}^{[p_1,\ldots,p_N]_S}_{(r+1/2)} = -\left(\prod_{i=1}^{N}\frac{1}{p_i!}\right)\int d^Dx\,\bar{\psi}^{\,\mu^1_1\ldots\mu^1_{p_1}\ldots\mu^i_1\ldots\mu^i_{p_i}\ldots\mu^N_{p_N}}\,\partial\!\!\!/\,\psi_{\mu^1_1\ldots\mu^1_{p_1}\ldots\mu^i_1\ldots\mu^i_{p_i}\ldots\mu^N_{p_N}} \tag{5.14}$$

on imposing the physical gauge conditions

$$d^{\dagger\,(i)}\psi = 0\,, \qquad \gamma^{\mu^i_1}\psi_{\mu^1_1\ldots\mu^1_{p_1}\ldots\mu^i_1\ldots\mu^i_{p_i}\ldots\mu^N_{p_N}} = 0 \tag{5.15}$$

for all $i = 1,\ldots,N$. Note that the second of these constraints implies $\tau^{(ij)}\psi = 0$ for any $i,j = 1,\ldots,N$. This follows by multiplying the gamma-tracelessness condition by $\gamma^{\mu^j_1}$ and using $\sigma^{(ij)}\psi = 0$ for $j > i$ (since the tensor part of ψ is $GL(D,\mathbb{R})$-irreducible).

To conclude, we have

$$\partial\!\!\!/\,\mathcal{G}^{(m)}_S(\psi) = \mathcal{G}^{(m)}(\psi) \tag{5.16}$$

as an operator equation for any spinor-valued fermionic tensor field ψ, where $\mathcal{G}^{(m)}(\psi)$ correspond to the second order operators defined in (3.2), (3.3), (3.6) and (3.7) but now acting on ψ. This generalises the result in [11]. Conversely,

$$\mathcal{G}^{(m)}_S(\psi) = \frac{\partial\!\!\!/}{\Box}\mathcal{G}^{(m)}(\psi)\,. \tag{5.17}$$

A. Fixing to physical gauge for higher spins

A free massless gauge field in D dimensions can be reduced, by gauge fixing and using the field equations, to the dynamical degrees of freedom corresponding to a field in a representation of the little group $\mathrm{SO}(D-2) \subset \mathrm{SO}(D-1,1)$ satisfying a free field equation. Rather than fully fixing such a light-cone gauge, it will be sufficient here to consider the analogue of the transverse traceless gauge in general relativity, and we shall refer to such gauges as 'physical gauges'. For example, consider a free massless totally symmetric tensor gauge field $\phi_{\mu_1\ldots\mu_s}$ of rank s. The gauge symmetry can be used to impose a gauge condition such as $\partial^{\mu_1}\phi_{\mu_1\mu_2\ldots\mu_s} = 0$ off-shell. However, if $\phi_{\mu_1\ldots\mu_s}$ satisfies its field equation, further restricted gauge transformations are possible while preserving the gauge conditions and these can be used to make $\phi_{\mu_1\ldots\mu_s}$ traceless, and the field equation then reduces to the free one $\Box\,\phi_{\mu_1\ldots\mu_s} = 0$. Then $\phi_{\mu_1\ldots\mu_s}$ is in physical gauge if

$$\Box\,\phi_{\mu_1\ldots\mu_s} = 0\,, \qquad \partial^{\mu_1}\phi_{\mu_1\mu_2\ldots\mu_s} = 0\,, \qquad \eta^{\mu_1\mu_2}\phi_{\mu_1\mu_2\ldots\mu_s} = 0 \tag{A.1}$$

on-shell, where $\eta^{\mu\nu}$ is the (inverse) $\mathrm{SO}(D-1,1)$-invariant metric. We now discuss this gauge-fixing in more detail for the examples $s = 2,3,4$.

A massless field with $s = 2$ describes a linearised graviton $h_{\mu\nu}$ whose field equation is the linearised Einstein equation

$$\mathcal{G}_{\mu\nu} := \Box h_{\mu\nu} - 2\partial^\rho\partial_{(\mu}h_{\nu)\rho} + \partial_\mu\partial_\nu h' = 0 \tag{A.2}$$

where $h' := \eta^{\mu\nu}h_{\mu\nu}$ is the trace. (A.2) is invariant under the gauge transformation

$$\begin{aligned}\delta h_{\mu\nu} &= 2\partial_{(\mu}\xi_{\nu)}\\ \delta h' &= 2\partial^\mu\xi_\mu\end{aligned} \tag{A.3}$$

for any one-form parameter ξ_μ. The De Donder gauge choice

$$T_\mu := \partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h' = 0 \tag{A.4}$$

uses D gauge symmetries to impose D constraints, but doesn't quite fix all the gauge symmetry, as it allows gauge transformations preserving the constraint $T_\mu = 0$,

$$\delta T_\mu = \Box \xi_\mu = 0 \tag{A.5}$$

restricting the gauge transformations to those with parameters ξ_μ satisfying $\Box\xi_\mu = 0$. For on-shell configurations satisfying (A.2), this residual symmetry can be used to eliminate the trace of the on-shell graviton h'. The field equation (A.2) implies $\Box h' = 0$ (using (A.4)) and for h' satisfying this, one can solve the equation

$$h' = 2\partial^\mu \zeta_\mu \tag{A.6}$$

for some ζ_μ satisfying $\Box\zeta_\mu = 0$ (see e.g. [21]), and so a gauge transformation with parameter $\xi_\mu = -\zeta_\mu$ can be used to set $h' = 0$ on-shell. Setting $h' = 0$ implies that (A.4) reduces to $\partial^\mu h_{\mu\nu} = 0$ and (A.2) reduces to $\Box h_{\mu\nu} = 0$, and the transverse traceless or physical gauge is achieved.

For $s = 3$, the field equation for a massless gauge field $\phi_{\mu\nu\rho}$ is

$$\mathcal{G}_{\mu\nu\rho} := \Box\phi_{\mu\nu\rho} - 3\partial^\alpha\partial_{(\mu}\phi_{\nu\rho)\alpha} + \partial_{(\mu}\partial_\nu\phi'_{\rho)} + \frac{1}{\Box}\left(2\partial^\alpha\partial^\beta\partial_{(\mu}\partial_\nu\phi_{\rho)\alpha\beta} - \partial_\mu\partial_\nu\partial_\rho\partial^\alpha\phi'_\alpha\right) = 0 \tag{A.7}$$

where $\phi'_\mu := \eta^{\nu\rho}\phi_{\mu\nu\rho}$ is the trace. This non-local equation is invariant under the gauge transformation

$$\begin{aligned} \delta\phi_{\mu\nu\rho} &= 3\partial_{(\mu}\xi_{\nu\rho)} \\ \delta\phi'_\mu &= \partial_\mu\xi' + 2\partial^\nu\xi_{\nu\mu} \end{aligned} \tag{A.8}$$

for any second rank symmetric tensor parameter $\xi_{\mu\nu}$. A convenient gauge choice is

$$T_{\mu\nu} := \partial^\rho\phi_{\mu\nu\rho} - \partial_{(\mu}\phi'_{\nu)} = 0 \tag{A.9}$$

but this still allows gauge transformations with parameters $\xi_{\mu\nu}$ satisfying

$$\delta T_{\mu\nu} = \Box\xi_{\mu\nu} - \partial_\mu\partial_\nu\xi' = 0\,. \tag{A.10}$$

These can now be used to eliminate the trace ϕ'_μ provided $\phi_{\mu\nu\rho}$ satsfies the field equation (A.7). The field equation (A.7) and gauge condition (A.9) imply that the trace satisfies

$$\Box\phi'_\mu = \partial_\mu\partial^\nu\phi'_\nu\,. \tag{A.11}$$

Given a second rank symmetric tensor $\zeta_{\mu\nu}$ which satisfies

$$\phi'_\mu = \partial_\mu\zeta' + 2\partial^\nu\zeta_{\nu\mu} \tag{A.12}$$

together with

$$\Box\zeta_{\mu\nu} - \partial_\mu\partial_\nu\zeta' = 0 \tag{A.13}$$

one can can perform a gauge transformation with parameter $\xi_{\mu\nu} = -\zeta_{\mu\nu}$ to set $\phi'_\mu = 0$, so that (A.9) reduces to $\partial^\mu\phi_{\mu\nu\rho} = 0$ and (A.7) reduces to $\Box\phi_{\mu\nu\rho} = 0$.

It remains to show that a tensor $\zeta_{\mu\nu}$ can be chosen to satisfy (A.12), (A.13). Define

$$f_\mu := \phi'_\mu - \partial_\mu\zeta' - 2\partial^\nu\zeta_{\nu\mu}\,. \tag{A.14}$$

The strategy, following [21], is to arrange for f_μ and $\dot{f}_\mu$ to vanish on an initial value surface $t = t_0$ (where $t := x^0$ is the time coordinate and $\dot{g} := \partial_t g$ for any tensor g). Then if $\Box f_\mu = 0$, f_μ will vanish everywhere and (A.12) will hold. Note that the trace of (A.13) vanishes identically, so that no constraint is imposed on $\Box\zeta'$ by (A.13). Then ζ' can be chosen to satisfy

$$\Box\zeta' = \frac{1}{3}\partial^\mu\phi'_\mu \tag{A.15}$$

so that this and (A.11) imply $\Box f_\mu = 0$ (using (A.13)), so that f_μ is harmonic when ϕ'_μ is on-shell. Then (A.13) becomes the following constraint on $\hat{\zeta}_{\mu\nu}$, the trace-free part of $\zeta_{\mu\nu}$,

$$\Box\hat{\zeta}_{\mu\nu} - \partial_\mu\partial_\nu\zeta' = -\frac{1}{D}\eta_{\mu\nu}\partial^\rho\phi'_\rho\,. \tag{A.16}$$

On the initial value surface $t = t_0$ we choose $\zeta_{\mu\nu}$, $\dot{\zeta}_{\mu\nu}$ to satisfy

$$\phi'_\mu = \partial_\mu\zeta' + 2\partial^\nu\zeta_{\nu\mu} \tag{A.17}$$

and

$$\dot{\phi}'_\mu = 3\partial_\mu\dot{\zeta}' + 2\nabla^i\dot{\zeta}_{i\mu} - 2\nabla^2\zeta_{0\mu} \tag{A.18}$$

where x^i with $i = 1,\ldots,D-1$ are the spatial coordinates and $\nabla^2 := \nabla^i\nabla_i$. These ensure that $f_\mu = 0$ and $\dot{f}_\mu = 0$ on $t = t_0$. Then $\zeta_{\mu\nu}$ is chosen to satisfy (A.13) and (A.16); given the initial values of $\zeta_{\mu\nu}$, $\dot{\zeta}_{\mu\nu}$ at $t = t_0$, this determines $\zeta_{\mu\nu}$ uniquely. Then as $\Box f_\mu = 0$, it follows that $f_\mu = 0$ everywhere, and as a result (A.12) and (A.13) are indeed satisfied, as required.

For $s = 4$, a massless field $\phi_{\mu\nu\rho\sigma}$ can satisfy the field equation

$$\begin{aligned}\mathcal{G}_{\mu\nu\rho\sigma} := {} & \Box\phi_{\mu\nu\rho\sigma} - 4\partial^\alpha\partial_{(\mu}\phi_{\nu\rho\sigma)\alpha} + 2\partial_{(\mu}\partial_\nu\phi'_{\rho\sigma)} + \\ & + \frac{1}{\Box}\left(4\partial^\alpha\partial^\beta\partial_{(\mu}\partial_\nu\phi_{\rho\sigma)\alpha\beta} - 4\partial_{(\mu}\partial_\nu\partial_\rho\partial^\alpha\phi'_{\sigma)\alpha} + \partial_\mu\partial_\nu\partial_\rho\partial_\sigma\phi''\right) = 0\end{aligned} \tag{A.19}$$

where $\phi'_{\mu\nu} := \eta^{\rho\sigma}\phi_{\mu\nu\rho\sigma}$ and $\phi'' := \eta^{\mu\nu}\phi'_{\mu\nu}$ are the single and double traces. The non-locality in (A.19) is again necessary so that it is invariant under the gauge transformation

$$\begin{aligned}\delta\phi_{\mu\nu\rho\sigma} &= 4\partial_{(\mu}\xi_{\nu\rho\sigma)} \\ \delta\phi'_{\mu\nu} &= 2\partial_{(\mu}\xi'_{\nu)} + 2\partial^\rho\xi_{\mu\nu\rho} \\ \delta\phi'' &= 4\partial^\mu\xi'_\mu\end{aligned} \tag{A.20}$$

with unconstrained third rank totally symmetric tensor parameter $\xi_{\mu\nu\rho}$. The gauge constraint

$$T_{\mu\nu\rho} := \partial^\sigma \phi_{\mu\nu\rho\sigma} - \frac{3}{2}\partial_{(\mu}\phi'_{\nu\rho)} = 0 \tag{A.21}$$

restricts the gauge transformations to those with parameters $\xi_{\mu\nu\rho}$ satisfying

$$\delta T_{\mu\nu\rho} = \Box \xi_{\mu\nu\rho} - 3\partial_{(\mu}\partial_\nu \xi'_{\rho)} = 0\,. \tag{A.22}$$

These can be used to set the single trace $\phi'_{\mu\nu}$ to zero when $\phi_{\mu\nu\rho\sigma}$ satisfies its field equations. The field equation (A.19) and gauge condition (A.21) imply that the trace satisfies

$$\Box \phi'_{\mu\nu} = 2\partial^\rho \partial_{(\mu}\phi'_{\nu)\rho} \tag{A.23}$$

and that

$$\partial_\mu \phi'' = 0\,, \qquad \partial^\mu \partial^\nu \phi'_{\mu\nu} = 0\,. \tag{A.24}$$

A third rank totally symmetric tensor $\zeta_{\mu\nu\rho}$ satisfying

$$\phi'_{\mu\nu} = 2\partial_{(\mu}\zeta'_{\nu)} + 2\partial^\rho \zeta_{\mu\nu\rho} \tag{A.25}$$

and

$$\Box \zeta_{\mu\nu\rho} - 3\partial_{(\mu}\partial_\nu \zeta'_{\rho)} = 0 \tag{A.26}$$

can then be used as a parameter of a gauge transformation with $\xi_{\mu\nu\rho} = -\zeta_{\mu\nu\rho}$ that sets $\phi'_{\mu\nu} = 0$, so that (A.21) reduces to $\partial^\mu \phi_{\mu\nu\rho\sigma} = 0$ and (A.19) reduces to $\Box \phi_{\mu\nu\rho\sigma} = 0$.

As before, such a tensor $\zeta_{\mu\nu\rho}$ can be found by first specifying $\zeta_{\mu\nu\rho}$, $\dot{\zeta}_{\mu\nu\rho}$ on an initial value surface $t = t_0$ and then using a wave equation for $\zeta_{\mu\nu\rho}$ to fix the tensor everywhere. Note that the trace of (A.26) does not restrict $\Box \zeta'_\mu$, but does imply

$$\partial_\mu \partial^\nu \zeta'_\nu = 0\,. \tag{A.27}$$

Equation (A.25) implies

$$\phi'' = 4\partial^\nu \zeta'_\nu \tag{A.28}$$

and both sides of this equation are constant, as a result of (A.24), (A.27). Defining

$$f_{\mu\nu} := \phi'_{\mu\nu} - 2\partial_{(\mu}\zeta'_{\nu)} - 2\partial^\rho \zeta_{\mu\nu\rho} \tag{A.29}$$

then if ζ'_μ is chosen to satisfy

$$\Box \zeta'_\mu = \frac{1}{3}\partial^\nu \phi'_{\mu\nu} \tag{A.30}$$

it follows that $\Box f_{\mu\nu} = 0$ on-shell. As above, $\zeta_{\mu\nu\rho}$, $\dot{\zeta}_{\mu\nu\rho}$ can be chosen at $t = t_0$ so that $f_{\mu\nu} = 0$ and $\dot{f}_{\mu\nu} = 0$ at $t = t_0$, and these together with (A.26), (A.30) determine $\zeta_{\mu\nu\rho}$ everywhere. It then follows that $f_{\mu\nu} = 0$ everywhere and so (A.25) and (A.26) are satisfied, as required.

B. The generalised Einstein tensor for type $[p_1, \ldots, p_N]$ gauge fields

As has been seen, the quadratic action for a type $[p_1, \ldots, p_N]$ gauge field is naturally written in terms of a generalised Einstein tensor E which is a gauge-invariant type $[p_1, \ldots, p_N]$ tensor that is conserved (i.e. $d^{\dagger(i)} E \equiv 0$ for all $i = 1, \ldots, N$). It is straightforward to construct E in simple examples such as those discussed in the introduction. In this appendix we use the multi-form structure to write an explicit form for E in some simple cases and the leading terms in the general case. Based on these results we propose an expression for the form of E in the general case.

Before discussing the construction for general N, it will prove useful to present the details for the $N = 2$ case. This class of bi-form gauge theories is illustrative of the general structure. In this case, we can use the simplified notation of [9, 10], dropping the superscript (1) and replacing the the superscript (2) with a tilde, so that $d := d^{(1)}$, $\tilde{d} := d^{(2)}$, and omitting the superscript (12), so that, e.g. $\tau := \tau^{(12)}$. We begin by noting the following identities for bi-form operators acting on a general element $T \in X^{p,q}$

$$
\begin{aligned}
d\tau^n + (-1)^{n+1}\tau^n d &= n\, \tilde{d}^\dagger \tau^{n-1} \\
d^\dagger \tau + \tau d^\dagger &= 0 \\
d\eta + \eta d &= 0 \\
d^\dagger \eta^n + (-1)^{n+1}\eta^n d^\dagger &= n\, \tilde{d}\eta^{n-1} \\
\tau\eta - \eta\tau &= (D - p - q)1 \\
\sigma\tau &= \tau\sigma \\
\sigma\eta &= \eta\sigma
\end{aligned}
\tag{B.1}
$$

with similar relations holding for the operators with tildes.

The general form for E can be written as

$$
E = \sum_{n=0}^{q} k_n\, \eta^n \tau^{n+1} F \tag{B.2}
$$

for some coefficients k_n, and these coefficients are fixed by requiring that E be conserved. For a gauge field $A \in X^{[p,q]}$ (with $p \geq q$) with field strength $F = d\tilde{d}A \in X^{[p+1,q+1]}$, the identities (B.1) imply that $\sigma\eta^m\tau^n F = 0$ for any powers m and n (since $\sigma F = 0$). This implies that each term in the sum (B.2) is annihilated by σ and so as a result each term in the sum is $GL(D, \mathbb{R})$-irreducible, in the $[p, q]$ representation. The identity

$$
d^\dagger(\eta^n\tau^n\mathcal{G}) = \eta^n\tau^n d^\dagger\mathcal{G} + n(n+1)\,\eta^{n-1}\tau^{n-1}d^\dagger\mathcal{G} \tag{B.3}
$$

allows the coefficients in (B.2) to be determined order by order in the expansion in the powers of η by requiring conservation of E at each order. Requiring that E satisfies $d^\dagger E \equiv 0$ and $\tilde{d}^\dagger E \equiv 0$ identically fixes the coefficients, giving the result

$$
E = \sum_{n=0}^{q} \frac{(-1)^n}{(n+1)(n!)^2}\, \eta^n \tau^{n+1} F \tag{B.4}
$$

JHEP05(2003)019

and E is in the $[p, q]$ representation, as required. The generalised Einstein equation $E = 0$ implies $\mathcal{G} := \tau F = 0$ for $D > p + q$. For example, for linearised gravity, $p = q = 1$ and the usual Einstein equation $E_{\mu\nu} = 0$ implies that the Ricci tensor vanishes in dimensions $D > 2$, but in the critical dimension $D = 2$, $E_{\mu\nu} = 0$ is an identity implying no restriction on the Ricci tensor.

We now turn to the general case of multi-form gauge fields $A \in X^{[p_1,\ldots,p_N]}$. The following identities for operators acting on a general element $T \in X^{p_1,\ldots,p_N}$ will be useful

$$\begin{aligned}
d^{(j)}\tau^{(ij)} + \tau^{(ij)}d^{(j)} &= d^{\dagger(i)} \\
d^{\dagger(j)}\tau^{(ij)} + \tau^{(ij)}d^{\dagger(j)} &= 0 \\
d^{(j)}\eta^{(ij)} + \eta^{(ij)}d^{(j)} &= 0 \\
d^{\dagger(j)}\eta^{(ij)} + \eta^{(ij)}d^{\dagger(j)} &= d^{(i)} \\
\tau^{(ij)}\tau^{(kj)} + \tau^{(kj)}\tau^{(ij)} &= 0 \\
\eta^{(ij)}\eta^{(kj)} + \eta^{(kj)}\eta^{(ij)} &= 0 \\
\sigma^{(ij)}\tau^{(ij)} - \tau^{(ij)}\sigma^{(ij)} &= 0 \\
\sigma^{(ij)}\tau^{(jk)} + \tau^{(jk)}\sigma^{(ij)} &= 0 \\
\sigma^{(ji)}\tau^{(jk)} + \tau^{(jk)}\sigma^{(ji)} &= -\tau^{(ik)} \\
\sigma^{(ij)}\eta^{(ij)} - \eta^{(ij)}\sigma^{(ij)} &= 0 \\
\sigma^{(ij)}\eta^{(jk)} + \eta^{(jk)}\sigma^{(ij)} &= -\eta^{(ik)} \\
\sigma^{(ji)}\eta^{(jk)} + \eta^{(jk)}\sigma^{(ji)} &= 0
\end{aligned} \tag{B.5}$$

Operators with distinct labels commute. Repeated labels are not to be summed.

The field equation for even N was given by (2.21) (or (3.2)) involving multiple traces of F, while that for odd N was formally very similar, given by (2.20) (or (3.3)) but with F replaced by the tensor ∂F, corresponding to a tableau with an even number $(N + 1)$ of columns. Below we will discuss the case of even N; similar formulae can be used for the odd N case provided F is replaced by ∂F.

Theorem. *For a gauge field $A \in X^{[p_1,\ldots,p_N]}$ (with $p_i \geq p_{i+1}$) with field strength $F = d^{(1)} \ldots d^{(N)}A \in X^{[p_1+1,\ldots,p_N+1]}$, the identities (B.5) imply that*

$$\sigma^{(ij)} \sum_{I \in S_N} \tau^{(i_1 i_2)} \ldots \tau^{(i_{N-1} i_N)} F = 0 \tag{B.6}$$

since $\sigma^{(ij)}F = 0$ for any $j > i$. The sum is over all values of $I = (i_1, \ldots, i_N)$ in $(1, \ldots, N)$ with no i_k equal. There are

$$f_N := \frac{N!}{2^{N/2}\,(N/2)!} \tag{B.7}$$

such inequivalent terms.[3]

[3]Two terms are said to be inequivalent if their indices cannot be rearranged such that they are proportional to each other.

Proof. Begin by partitioning the sum in (B.6) into two seperate sums for any given $\sigma^{(ij)}$. The first sum $\mathcal{G}_1$ contains f_{N-2} inequivalent terms whose elements each have one $\tau^{(ij)}$ in the $(N/2)$-fold trace of F. The second sum $\mathcal{G}_2$ contains the remaining $f_N - f_{N-2}$ inequivalent terms whose elements each have one $\tau^{(ii_r)}$ and one $\tau^{(ji_s)}$ where i_r and i_s are different labels for different terms but never equal i nor j. Since each of the $N/2$ traces have different labels then they commute and can be arbitrarily permuted. We therefore choose each $\tau^{(ji_s)}$ to be leftmost in $\mathcal{G}_2$. From (B.5) it is clear that $\sigma^{(ij)}$ commutes with all the traces in $\mathcal{G}_1$. Moreover, $\sigma^{(ij)}$ commutes with all traces in $\mathcal{G}_2$ except the two $\tau^{(ji_s)}$ and $\tau^{(ii_r)}$ traces. (B.5) shows that $\sigma^{(ij)}$ anticommutes with the first of these traces $\tau^{(ji_s)}$ then anticommutes with the second $\tau^{(ii_r)}$ but also produces another term with $\tau^{(ji_r)}$ replacing each $\tau^{(ii_r)}$. More precisely this means

$$\begin{aligned}\sigma^{(ij)} \sum_{I\in S_N} \tau^{(i_1 i_2)} \dots \tau^{(i_{N-1} i_N)} F - \sum_{I\in S_N} \tau^{(i_1 i_2)} \dots \tau^{(i_{N-1} i_N)} \sigma^{(ij)} F = \\ = \sum \tau^{(ji_s)} \tau^{(ji_r)} \tau^{(i_1 i_2)} \dots \tau^{(i_{N-1} i_N)} F\end{aligned} \tag{B.8}$$

where the sum on the right hand side is still over all labels $i_k \neq i, j$ with $k = 1, \dots, s, \dots, r, \dots, N$. However, the first two traces in this sum have a common index j and therefore anticommute (using (B.5)). Since one sums over all inequivalent labels then for each $i_s = m$ and $i_r = n$, there will be a corresponding pair $i_s = n$ and $i_r = m$ and so this sum is identically zero. Using that F is irreducible under $GL(D, \mathbb{R})$ then completes the proof. □

The $GL(D, \mathbb{R})$-irreducible term

$$\mathcal{G} := \sum_{I\in S_N} \tau^{(i_1 i_2)} \dots \tau^{(i_{N-1} i_N)} F \tag{B.9}$$

is the leading term in $E \in X^{[p_1,\dots,p_N]}$. The first shifted trace term is given by

$$\eta \cdot \tau \mathcal{G} := \mathcal{Y}_{[p_1,\dots,p_N]} \circ \left(\frac{1}{2} \sum_{i,j=1}^{N} \eta^{(ij)} \tau^{(ij)} \right) \mathcal{G} \tag{B.10}$$

with relative coefficient $k_1 = -1/N$ chosen to ensure conservation of $E = \mathcal{G} - (1/N)\eta \cdot \tau\mathcal{G} + \dots$ to first order. This coefficient is computed using the relation

$$d^{\dagger(i)} (\eta \cdot \tau \mathcal{G}) = \eta \cdot \tau \, d^{\dagger(i)} \mathcal{G} + N \, d^{\dagger(i)} \mathcal{G} \, . \tag{B.11}$$

A natural guess for the next shifted trace term is that it should be proportional to

$$\eta^2 \cdot \tau^2 \mathcal{G} := \mathcal{Y}_{[p_1,\dots,p_N]} \circ \left(\frac{1}{4} \sum_{i,j,k,l=1}^{N} \left(\eta^{(ij)} \eta^{(kl)} + 2\eta^{(ik)} \eta^{(lj)} \right) \tau^{(ij)} \tau^{(kl)} \right) \mathcal{G} \, . \tag{B.12}$$

Similarly, a natural proposal for the general n-trace correction term is that it be proportional to

$$\begin{aligned}\eta^n \cdot \tau^n \mathcal{G} := \mathcal{Y}_{[p_1,\dots,p_N]} \circ \\ \circ \left(\frac{1}{2^n} \sum_{i_1,\dots,i_{2n}=1}^{N} \left(\sum_{\pi} \eta^{(\pi(i_1)\pi(i_2))} \dots \eta^{(\pi(i_{2n-1})\pi(i_{2n}))} \right) \tau^{(i_1 i_2)} \dots \tau^{(i_{2n-1} i_{2n})} \right) \mathcal{G} \, .\end{aligned} \tag{B.13}$$

The bracketed sum in (B.13) is over all permutations of labels $(i_1, \ldots, i_{2n})$. Those permutations which occur more than once should be counted with multiplicity (hence the factor of 2 in the second term in (B.12)). In general, an explicit Young projection onto the $[p_1, \ldots, p_N]$ representation is required, although it can be the case, as in the bi-form examples above, that it is not needed as the term is already irreducible without projection.

This leads to the conjecture that E is given by

$$E = \sum_n k_n \, \eta^n \cdot \tau^n \mathcal{G} \tag{B.14}$$

with coefficients k_n chosen so that $d^{\dagger(i)} E \equiv 0$. It is clear, however, that the structure of the higher trace correction terms is complicated and this makes the determination of coefficients k_n difficult for $n > 1$.

The ordering of algebraic operators in (B.14) (with all η's to the left) is chosen because $d^{\dagger(i)}(\eta^n \cdot \tau^n \mathcal{G})$ can be expressed as terms with only n and $n-1$ traces of $d^{\dagger(i)}\mathcal{G}$ (rather than a sum of terms with all possible traces). This follows from the identities (B.5). If our conjectured form for the shifted traces is correct, then it is to be expected that there should be a general relation

$$d^{\dagger(i)}(\eta^n \cdot \tau^n \mathcal{G}) = \eta^n \cdot \tau^n \, d^{\dagger(i)}\mathcal{G} + \mathsf{P}_N(n) \, \eta^{n-1} \cdot \tau^{n-1} \, d^{\dagger(i)}\mathcal{G} \tag{B.15}$$

for some polynomial $\mathsf{P}_N(n)$ in N and n. This structure was found above for $\mathsf{P}_N(1) = N$ (B.11) and $\mathsf{P}_2(n) = n(n+1)$ (B.3) and will be checked in further examples below. If (B.15) is true then conservation of E implies

$$k_n = (-1)^n \left(\prod_{r=1}^{n} \mathsf{P}_N(r) \right)^{-1} \tag{B.16}$$

after choosing $k_0 = 1$.

A check of the conjecture (B.15) is given by considering the example of totally symmetric rank s gauge fields. We again take s to be even; the odd spin case is similar but with F replaced by ∂F. As noted in the introduction, the gauge-invariant field strength F for a spin-s (type $[1, \ldots, 1]$ tensor) gauge field is a type $[2, \ldots, 2]$ tensor. The corresponding gauge-invariant field equation is defined using (B.9) and is given by the vanishing of the spin-s tensor $\mathcal{G}$ with components

$$\mathcal{G}_{\mu_1 \ldots \mu_s} := f_s \, \eta^{(\nu_1 \nu_2} \ldots \eta^{\nu_{s-1} \nu_s)} F_{\mu_1 \nu_1 \ldots \mu_s \nu_s} \tag{B.17}$$

where f_s is the numerical factor defined in (B.7) so that each inequivalent term in (B.17) contributes with weight 1. The shifted trace terms are defined using (B.13) and are also spin-s tensors whose components are

$$(\eta^n \cdot \tau^n \mathcal{G})_{\mu_1 \ldots \mu_s} := g_{s,n} \, \eta_{(\mu_1 \mu_2} \ldots \eta_{\mu_{2n-1} \mu_{2n}} \, \mathcal{G}^{\{n\}}_{\mu_{2n+1} \ldots \mu_s)} \tag{B.18}$$

where $\mathcal{G}^{\{n\}} := \tau^n \mathcal{G}$ is the nth trace of $\mathcal{G}$ for $0 \leq n \leq s/2$. There is just a single inequivalent trace τ on totally symmetric tensors. The numerical factor

$$g_{s,n} := \frac{s!}{n! \, 2^n \, (s-2n)!} \tag{B.19}$$

ensures that each inequivalent term in (B.18) contributes with weight 1. The shifted trace terms (B.18) for spin-s gauge fields are irreducible, without the need for an explicit Young projection — in contrast to the general case discussed above. This is due to the definition of the sum over permutations in (B.13) which is equivalent to a total symmetrisation of indices when acting on a totally symmetric tensor.

Using the relation

$$\partial^{\mu} \mathcal{G}^{\{n-1\}}_{\mu\mu_{2n}\ldots\mu_s} = \frac{(s+1-2n)}{(s+2-2n)}\, \partial_{(\mu_{2n}}\, \mathcal{G}^{\{n\}}_{\mu_{2n+1}\ldots\mu_s)} \tag{B.20}$$

then implies

$$d^{\dagger}(\eta^n \cdot \tau^n \mathcal{G}) = \eta^n \cdot \tau^n\, d^{\dagger}\mathcal{G} + \mathsf{P}_s(n)\, \eta^{n-1} \cdot \tau^{n-1}\, d^{\dagger}\mathcal{G} \tag{B.21}$$

where $\mathsf{P}_s(0) = 0$ and $\mathsf{P}_s(n) = s + 2 - 2n$ for $0 < n \leq s/2$. These numbers agree with those found above for $\mathsf{P}_N(n)$ for those values of (N, n) for which they are both defined. That is, one finds $\mathsf{P}_s(1) = s$ from either (B.21) or from (B.11) and (B.3). One can then solve for the spin-s Einsten tensor with even s in the manner described above, so that

$$E = \mathcal{G} + \sum_{n=1}^{s/2} \frac{(-1)^n}{\left(\prod_{r=1}^{n}(s+2-2r)\right)}\, \eta^n \cdot \tau^n \mathcal{G} \tag{B.22}$$

which satisfies the single inequivalent conservation condition $d^{\dagger}E \equiv 0$ identically.

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JHEP05(2003)019

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JHEP05(2003)019